A K-theoretical invariant and bifurcation for a parameterized family of functionals

Alessandro Portaluri *

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Abstract

Let $\mathscr{F} := \{f_x : x \in X\}$ be a family of functionals defined on a Hilbert manifold $\widetilde{E}$ and smoothly depending on a compact n-dimensional manifold X. The aim of this paper is to give a sufficient topological condition on the parameter space X in such a way the family $\mathscr{F}$ bifurcate from the trivial branch.

Introduction

The meaning of the word *bifurcation* changes depending on the context and here we deal only with bifurcation from the *trivial branch*. This is perhaps the simplest bifurcation phenomena which occur in nonlinear analysis, differential geometry and mechanics. The aim of this paper is to establish an abstract bifurcation result in a form suitable for the applications to geometric variational problems.

Given two normed linear spaces E and F and a continuous family of Fredholm maps $h \colon X \times E \to F$ continuously parameterized by a path connected topological space such that $h(\cdot, 0) = 0$, we say that a point $x_* \in X$ is a *bifurcation point from the trivial branch $X \times \{0\}$* if each neighborhood of $(x_*, 0)$ contains solutions of $h(x, v) = 0$ with $v \neq 0$. We shall denote by $\mathrm{Bif}\,(h)$ the set of all bifurcation points of h. It is a well-known fact that there exists a strict relation between the topology of the parameter space and the existence of a bifurcation point from the trivial branch and thus an important role is played by the homotopy theoretical methods. The literature on this topic is quite broad and we will be able to review just some of the existing results.

Homotopic methods in several parameter bifurcation theory were developed in the past decades mainly by Alexander, Ize, Fitzpatrick, Pejsachowicz among others. The Alexander's approach can be sketched as follows. Consider a family $f_p : \mathbb{R}^n \to \mathbb{R}^n$ parameterized by $p \in \mathbb{R}^k$, with $f_p(0) = 0$ and let L_p be the derivative of f_p at 0. Let p_0 be an isolated point in the set $\Sigma(f)$ of all parameters at which the derivative L_p is singular. Restricting the map $p \mapsto L_p$ to the boundary of a small disk D centered at p_0 one gets an element γ_f in the homotopy group $\pi_{k-1}(\mathrm{Gl}(n))$. This element is an obstruction to the existence of a deformation of the map $L : (D, \partial D) \to (M_n(\mathbb{R}), \mathrm{Gl}(n))$ to a map into $\mathrm{Gl}(n)$. The point p_0 is a bifurcation point, provided the image of γ_f by the classical Whitehead J-homomorphism is nontrivial.

Although, completely different in principle, the Fitzpatrick and Pejsachowicz bifurcation invariant can be seen as a global version of the Alexander's invariant. In fact given a family of quasilinear Fredholm maps of index zero continuously parameterized by a smooth compact manifold X with $f_x(0) = 0$, the obstruction to deforming such a family to a family of isomorphism is the *analytical index* (or *index bundle*), namely an element of the reduced Grothendieck group of virtual vector bundles $\widetilde{KO}(X)$ and quite naturally the bifurcation invariant in this setting is

provided by $J(\text{Ind } L)$, where where now J is the J-homomorphism of Atiyah-Adams. Moreover in [3] it was proven that if f is such a family then there is a homotopy invariant belonging to the first (singular) cohomology group with $\mathbb{Z}_2$ coefficients whose non-triviality insures the existence of a bifurcation point from the trivial branch. This invariant is in fact the first Stiefel-Whitney class of the analytical index and its non-triviality can be geometrically interpreted as the non-orientability of the index bundle.

What we propose here is to consider the particular case of families of self-adjoint Fredholm maps which naturally arise in calculus of variation, by showing that in this situation a more refined invariant arises. In fact, we will show that the bifurcation is related to the non triviality of the *first Chern class of the index bundle*. Moreover, by using this invariant, we are able to prove a result about the *Lebesgue covering dimension* of the bifurcation set in the same vein of [3].

1 Spectral flow and Index bundle

The aim of this section is to briefly recall the basic definitions and properties about the spectral flow for paths of Fredholm quadratic forms. Furthermore we will construct the index bundle for a family of selfadjoint Fredholm operators, smoothly parameterized by a compact path-connected smooth manifold. Our main references will be the following [1, 2, 3, 4, 7, 9].

Spectral flow of Fredholm forms. Let $(\mathscr{H}, \langle \cdot, \cdot \rangle_{\mathscr{H}})$ be a complex separable Hilbert space.

DEFINITION 1.1 *A* Fredholm quadratic form *is a function* $q\colon \mathscr{H} \to \mathbb{R}$ *such that there exists a bounded symmetric sesquilinear form* $b := b_q\colon \mathscr{H} \times \mathscr{H} \to \mathbb{R}$ *with* $q(u) = b(u, u)$ *and with* $\ker b$ *of finite dimension.*

The space $\mathscr{Q}(\mathscr{H})$ of all bounded quadratic forms is a Banach space with the sup norm and the set $\mathscr{Q}_F(\mathscr{H})$ of all Fredholm quadratic forms is an open subset of $\mathscr{Q}(\mathscr{H})$ that is stable under perturbations by weakly continuous quadratic forms. Let $q\colon (S^1, s_0) \to (\mathscr{Q}_F(\mathscr{H}), q_0)$ be a based loop of Fredholm quadratic forms.

DEFINITION 1.2 *The* spectral flow of a loop $q\colon S^1 \to \mathscr{Q}_F(\mathscr{H})$ *of bounded Fredholm quadratic form is defined by*
$$\operatorname{sf}(q, S^1) := \operatorname{sf}(A_{q(t)}, S^1)$$
where $A_{q(t)}$ *is the unique bounded linear self-adjoint Fredholm operator such that*
$$\left\langle A_{q(t)} u, u \right\rangle_{\mathscr{H}} = q(t)(u) \qquad \text{for all } u \in \mathscr{H}.$$

Given any differentiable loop of Fredholm quadratic forms $q\colon S^1 \to \mathscr{Q}_F(\mathscr{H})$ then, for all $t \in S^1$, the derivative $\dot{q}(t)$ is also a quadratic form. We say that a point t is a *crossing point* if $\ker b_{q(t)} \neq \{0\}$ and we say that the crossing point t is *regular* if the *crossing form* $\Gamma(q, t)$, defined as the restriction of the derivative $\dot{q}(t)$ to the subspace $\ker b_{q(t)}$, is non-degenerate. It is easy to see that regular crossings are isolated and that the property of having only regular crossing forms is generic for paths in $\mathscr{Q}_F(\mathscr{H})$. From theorem 4.1 in [4] we deduce the following.

PROPOSITION 1.3 *If all crossing points of the path are regular then they are finite in number and*
$$\operatorname{sf}(q, S^1) = \sum_i \operatorname{sign} \Gamma(q, t_i). \tag{1}$$

Next, we define the spectral flow for a family of Fredholm quadratic forms defined on a separable Hilbert bundle.

DEFINITION 1.4 *A generalized family of Fredholm quadratic forms* parameterized by X *is a smooth function* $q\colon \widetilde{\mathscr{H}} \to \mathbb{R}$, *where* $\widetilde{\mathscr{H}}$ *is a complex separable Hilbert bundle over X and q is such that its restriction q_x to the fiber $\widetilde{\mathscr{H}_x}$ over x is a (bounded) Fredholm quadratic form.*

2

If $(X, x_0) = (S^1, s_0)$ and $q_{s_0} \in \mathrm{Gl}(\widetilde{\mathscr{H}_{s_0}})$, we define the spectral flow $\mathrm{sf}(q) := \mathrm{sf}(q, S^1)$ of such a family q, by choosing a trivialization

$$\Phi \colon S^1 \times \widetilde{\mathscr{H}_{s_0}} \to \widetilde{\mathscr{H}} \colon (t, u) \mapsto \Phi_t(u)$$

and by setting

$$\mathrm{sf}(q) = \mathrm{sf}(\widetilde{q}, S^1), \qquad \text{where} \quad \widetilde{q}(t)[u] := q_t(\Phi_t u). \tag{2}$$

From the cogredience property of the spectral flow the right hand side of (2) is independent of the choice of the trivialization.

DEFINITION 1.5 *Let X be a compact path-connected smooth n-dimensional manifold, A, B be two real (resp. complex) vector bundles over X and $\Theta^n = X \times \mathbb{R}^n$ be the trivial real ($\Theta^n = X \times \mathbb{C}^n$ resp. complex) vector bundle of rank n over X.*

1. *We say that A, B over X are* stably equivalent *if they are (bundle) isomorphic up addition of trivial bundles; namely $A \oplus \Theta^n \cong B \oplus \Theta^m$ where $n, m \in \mathbb{N}$.*

2. *If $\mathrm{Vect}_{\mathbb{R}}(X)$ (resp. $\mathrm{Vect}_{\mathbb{C}}(X)$) is the set of all isomorphism classes of real (resp. complex) vector bundles over X then we define the* (full) Grothendieck group $K(X)$ *as the group completion of the abelian semigroup $\mathrm{Vect}_{\mathbb{R}}(X)$ (resp. $\mathrm{Vect}_{\mathbb{C}}(X)$).*

3. *We term* reduced Grothendieck group of virtual bundles of rank 0 *and we denote it respectively by $\widetilde{KO}(X)$ (resp. $\widetilde{K}(X)$), the set of all stable equivalence classes of real (resp. complex) vector bundles over X.*

Thus each element in $KO(X)$ (resp. $K(X)$), can be written as difference $[A] - [B]$ where A, B are vector bundles over X and where the brackets $[\cdot]$ denotes the stable equivalence class. Thus the map which assign to each stable equivalence class of a real (resp. complex) vector bundle X the difference bundle $[A] - [\Theta^n]$ gives the inclusion between $\widetilde{KO}(X)$ in $KO(X)$ (resp. $\widetilde{K}(X)$ in $K(X)$). Furthermore, for any negative integer $-n$, with $n \in \mathbb{N}$, the group $K^{-n}(X)$ is defined as the K-group of all complex vector bundles over the n-times (unreduced) suspension of X

$$S^n X := (S^n \times X)/(S^n \vee X).$$

The Index bundle. Given a continuous path of bounded Fredholm operators $L \colon X \to \mathscr{F}(\mathscr{H})$ we can associate a homotopy invariant called *analytical index* or *index bundle* which is represented by a stable equivalence class of bundles as follows. By compactness of X, there exists a finite dimensional subspace V of $\mathscr{H}$ transverse to the family L meaning that

$$\mathrm{Im}\, L_x + V = \mathscr{H}, \qquad \forall\, x \in X.$$

Thus the set

$$Y = \{(x, v) \in X \times \mathscr{H} \colon \quad L_x v \in V\}$$

is the total space of a real (resp. complex) vector bundle over X with fiber $Y_x := L_x^{-1}(V)$.

DEFINITION 1.6 *We define the* analytical index *or the* index bundle *of the family the element $\mathrm{Ind}\,(L) \in \widetilde{KO}(X)$ (resp. in $\widetilde{K}(X)$) which represents the stable equivalence class of the real (resp. complex) vector bundle Y over X; namely:*

$$\mathrm{Ind}\,(L) = [Y] - [\Theta(V)],$$

where $[\Theta(V)]$ is the trivial bundle over X with fiber V.

REMARK 1.7 *We observe that it does not depend on the choice of the subspace V but just on the homotopy class of the family. (See for instance [11, Theorem 2.1]).*

Now in order to develop an index theory in the self-adjoint case, the main difficulty is due to the fact that the analytical index of the family vanishes. However it turns out that one can develop an analogous theory for the space $\mathscr{F}_S(\mathscr{H})$ of all real (resp. complex) bounded self-adjoint Fredholm operators not based on the *index* which vanishes on $\mathscr{F}_S(\mathscr{H})$ but on the concept of *spectral flow*.

PROPOSITION 1.8 *([1, Theorem B])*

1. *From the topological point of view the space $\mathscr{F}_S(\mathscr{H})$ has three connected components denoted by $\mathscr{F}_S^+(\mathscr{H})$, $\mathscr{F}_S^-(\mathscr{H})$ and $\mathscr{F}_S^i(\mathscr{H})$ respectively of all* essentially positive, negative and indefinite, *where, essentially positive (resp. negative) means that the spectrum is positive (resp. negative) on some invariant subspace of $\mathscr{H}$ of finite codimension.*

2. *The components $\mathscr{F}_S^+(\mathscr{H})$, $\mathscr{F}_S^-(\mathscr{H})$ both are contractible.*

3. *Define a map*

$$\alpha \colon \mathscr{F}_S^i(\mathscr{H}) \to \Omega\mathscr{F}(\mathscr{H})$$

where $\Omega\mathscr{F}(\mathscr{H})$ denotes the loop of the space of all Fredholm operators obtained by assigning to each $A \in \mathscr{F}_S^i(\mathscr{H})$ the path from Id *to* $-$Id *in $\mathscr{F}(\mathscr{H})$ given by*

$$\alpha(A) := \mathrm{Id}\cos(\pi t) + iA\sin(\pi t), \qquad t \in [0,1]$$

and closed to a loop by the standard continuation

$$\mathrm{Id}\left(\cos(\pi t) + i\sin(\pi t)\right), \qquad t \in]1,2].$$

Then α is a hommotopy equivalence, and so $\mathscr{F}_S^i(\mathscr{H})$ is a classifying space for the functor K^{-1}.

Moreover by taking into account the Atiyah-Jänich theorem, we get

$$K^{-1}(X) := K^0(SX) \simeq [SX, \mathscr{F}(\mathscr{H})] \simeq [X, \Omega\mathscr{F}(\mathscr{H})]. \tag{3}$$

DEFINITION 1.9 *Let $L \colon X \to \mathscr{F}_S^i(\mathscr{H})$ be a continuous family of (complex) bounded linear self-adjoint Fredholm operators parameterized by X. The* analytical index *of the family is the homotopy class of this map as an element of the group $K^{-1}(X)$ namely:*

$$\mathrm{Ind}_\alpha L := [\alpha \circ L] \in K^{-1}(X)$$

where α is the homotopy equivalence given above.

Denoting by c_1 the first Chern class and by taking into account its funtorial properties, it is readily seen that it depends only on the stable equivalence class of complex vector bundles and hence it induces a natural transformation of functors

$$c_1 \colon K^0(-) \longrightarrow H^2(-, \mathbb{Z}). \tag{4}$$

and therefore also on $\widetilde{K}$ by restriction. We denote by σ^K and σ^H the suspension isomorphisms in K-theory and in singular cohomology respectively and let us consider the commutative diagram below

$$
\begin{array}{ccc}
\widetilde{K}^0(SX) & \xrightarrow{\;c_1\;} & H^2(SX, \mathbb{Z}) \\
\sigma^K \big\uparrow & & \big\uparrow \sigma^H \\
\widetilde{K}^{-1}(X) & \longrightarrow & H^1(X, \mathbb{Z})
\end{array}
$$

By using the commutativity, we can define the map $\bar{c}_1 \colon \widetilde{K}^{-1}(X) \to H^1(X, \mathbb{Z})$, by setting

$$\bar{c}_1 := \sigma^H \circ c_1 \circ \sigma^K.$$

The following will be crucial in the sequel.

PROPOSITION 1.10 *([2, Corollary 1.13]) Let $L\colon S^1 \to \mathscr{F}_S^i(\mathscr{H})$ be a continuous path. Then we have:*

$$\mathrm{sf}_{\mathbb{C}}(L, S^1) = \bar{c}_1 \left(\mathrm{Ind}\,_\alpha L\right)[S^1]$$

where as usually $[\cdot]$ denotes the fundamental class.

Given the real separable Hilbert space $(\mathscr{H}, \langle \cdot, \cdot \rangle_{\mathscr{H}})$ and denoted by $(\mathscr{H}^{\mathbb{C}}, \langle \cdot, \cdot \rangle_{\mathscr{H}^{\mathbb{C}}})$ its complexification, for any continuous path of selfadjoint Fredholm operators $L\colon (S^1, s_0) \to \mathscr{F}_S(\mathscr{H})$ we will denote by $L^{\mathbb{C}}$ its complexification.

LEMMA 1.11 *Under the notations above, we have*

$$\mathrm{sf}_{\mathbb{C}}(L^{\mathbb{C}}, S^1) = 2\,\mathrm{sf}_{\mathbb{R}}(L, S^1).$$

Proof. At first we prove the thesis for C^1-paths of bounded self-adjoint Fredholm operators having only regular crossings. Once this is done we can conclude simply by invoking the homotopy invariance property of the spectral flow for admissible paths and the fact that each based loop of Fredholm operators is homotopy to a continuous loop having only regular crossings.

By Proposition 1.3 the spectral flow is computed by formula (1). Now since at each crossing point s_0, we have $\ker L_{s_0}^{\mathbb{C}} = \ker L_{s_0} \oplus i \ker L_{s_0}$ then

$$\dim_{\mathbb{R}}(\ker L_{s_0}^{\mathbb{C}}) = 2 \dim_{\mathbb{R}}(\ker L_{s_0}).$$

Furthermore, since the inertial indexes of the path $L^{\mathbb{C}}$ are equals to the inertial indexes of the path L, then at each crossing point s_0, the following equality holds

$$\mathrm{sign}\,\Gamma(L^{\mathbb{C}}, s_0) = \mathrm{sign}\,\Gamma(L, s_0).$$

Now the thesis follows by summing over all (finite) crossings. $\qquad\qquad q.e.d.$

2   The main results

The aim of this section is to state and prove the main result of this paper. First of all we need the following

DEFINITION 2.1 *A smooth family of Hilbert manifolds $\{E_x\}_{x \in X}$ parameterized by X and modeled on the separable Hilbert space E is a family of manifolds of the form $E_x = p^{-1}(x)$ where $p\colon \widetilde{E} \to X$ is a smooth submersion of a Hilbert manifold $\widetilde{E}$ onto X.*

By the implicit function theorem each fiber E_x of the submersion is a submanifold of X of finite codimension (more precisely the codimension is equal to the dimension of the compact manifold X). For each $e \in E_x$, the tangent space $T_e E_x$ coincides with $\ker \mathrm{d}p_e$. Being p a submersion the family of Hilbert vector spaces $\mathrm{TF}(p) = \{\ker \mathrm{d}p_x \colon x \in X\}$ is a Hilbert subbundle of the tangent bundle $T\widetilde{E}$. $\mathrm{TF}(p)$ is the bundle of tangents along the fibers or the vertical bundle of the submersion p.

Thus given a smooth functional $f\colon \widetilde{E} \to \mathbb{R}$, it defines by restriction to the fibers of p a *smooth family of functionals* $f_x\colon E_x \to \mathbb{R}$. We will assume that there exists a smooth section $\sigma\colon X \to \widetilde{E}$ of p such that $\sigma(x)$ is a critical point of the restriction f_x of the functional f to the fiber E_x and we will refer to σ as the *trivial branch* of critical points of the family $\{f_x \colon x \in X\}$.

DEFINITION 2.2 *A point $x_* \in X$ is a* bifurcation point *of the family $\{f_x \colon x \in X\}$ from the trivial branch $\sigma(X)$ if there exists a sequence $x_n \to x_*$ and a sequence $e_n \to \sigma(x_*)$ such that $p(e_n) = x_n$ and each e_n is a critical point of f_{x_n} not belonging to $\sigma(X)$.*

In what follows we shall denote by h_x the Hessian of f_x at the point $\sigma(x)$. Our next assumption is the following

(H) for each $x \in X$, the Hessian h_x is a self-adjoint Fredholm quadratic form.

The family of Hessians h_x, for $x \in X$, defines a smooth function h on the total space of the pull-back bundle $\sigma^* \mathrm{TF}(p)$ of the vertical bundle $\mathrm{TF}(p)$ by the map $\sigma \colon X \to \widetilde{E}$ such that the restriction of h to each fiber is a Fredholm quadratic form. The function $h \colon \sigma^* \mathrm{TF}(p) \to \mathbb{R}$ is a *generalized family of Fredholm quadratic forms.* According to the previous notation, we shall denote by $\widetilde{h}^{\mathbb{C}}$ the complex selfadjoint extension of the family $\widetilde{h}$ to the complexified Hilbert subbundle $(\sigma^* \mathrm{TF}(p))^{\mathbb{C}}$.

THEOREM 2.3 *Let X be a compact connected n-dimensional manifold, $p \colon \widetilde{E} \to X$ a smooth submersion of a real separable Hilbert manifold $\widetilde{E}$ over X modeled on E and $f \colon \widetilde{E} \to \mathbb{R}$ be a smooth function. Assume that:*

$$0 \neq \bar{c}_1(\mathrm{Ind}_\alpha \widetilde{h}^{\mathbb{C}}) \in H^1(X, \mathbb{Z}).$$

Then the Lebesgue covering dimension of $\mathrm{Bif}\,(f)$ is at least $n-1$. Moreover, either $\mathrm{Bif}\,(f)$ disconnects X or it is not contractible to a point.

Proof. First of all we observe that $B := \mathrm{Bif}\,(f)$ disconnects X then the dimension of B is at least $n-1$ since the complement of a closed subset of dimension less that $n-1$ is always connected. Therefore let us assume that $X \backslash B$ is connected.

By the vector bundle neighborhood theorem Theorem [7, Appendix A], there exist a trivial Hilbert bundle $\mathscr{E} := X \times E$ over X and a fiber preserving smooth map $\psi \colon \mathscr{E} \to \sigma^* \mathrm{TF}(p)$ such that $\psi(x, 0) = \sigma(x)$ and such that ψ is a diffeomorphism of $\mathscr{E}$ with an open neighborhood $\mathscr{O}$ of $\sigma(X)$ in $\sigma^* \mathrm{TF}(p)$. Let $\widetilde{f} \colon X \times E \to \mathbb{R}$ be defined by $\widetilde{f} = f \circ \psi$. Then $\widetilde{f}$ is a family of smooth functionals on the (real) Hilbert space E. Since the restriction ψ_x of ψ to the fiber is a diffeomorphism, we have that $u \in E$ is a critical point of $\widetilde{f}_x = f_x \circ \psi_x$ if and only if $\psi_x(u)$ is a critical point of f_x. In particular 0 is a critical point of $\widetilde{f}_x$ for each $x \in X$. The Hessian $\widetilde{h}_x$ of $\widetilde{f}_x$ at 0 is given by $\widetilde{h}_x(\xi) = h_x(d\psi_x(0))[\xi]$.

Now let $\widetilde{h}_x^{\mathbb{C}}$ be a self-adjoint extension of the Hessian quadratic form on the complexified Hilbert space $E^{\mathbb{C}}$, and, for each $x \in X$, let $L_x^{\mathbb{C}}$ be the self-adjoint Fredholm operator representing the Fredholm Hermitian form $\widetilde{h}_x^{\mathbb{C}}$. We denote by $\mathrm{Ind}_\alpha \widetilde{h}^{\mathbb{C}}$, the index bundle of the induced family $L^{\mathbb{C}}$ of selfadjoint Fredholm operators on the complex Hilbert space $E^{\mathbb{C}}$ parameterized by X, and let us consider its first Chern class $c := \bar{c}_1(\mathrm{Ind}_\alpha \widetilde{h}^{\mathbb{C}}) \in H^1(X, \mathbb{Z})$. Since by assumption $\bar{c}_1(\mathrm{Ind}_\alpha \widetilde{h}^{\mathbb{C}}) \neq 0$ then there must be some homology class $[\gamma] \in H^1(X, \mathbb{Z})$ such that the Kronecker pairing $\langle c, [\gamma] \rangle$ is different from 0. Let $\zeta \in H^{n-1}(X, \mathbb{Z})$ be Poincaré dual of $[\gamma]$.

To prove the statement we need to show that the restriction of this class to B gives arise a nontrivial class in $\overline{H}^{(n-1)}(B, \mathbb{Z})$. Here $\overline{H}^*(B, \mathbb{Z}) = \mathrm{dirlim}\, H^*(U, \mathbb{Z})$ where U ranges over all neighborhoods of B in X. Consider the commutative diagram:

$$
\begin{array}{ccc}
H^{n-1}(X, \mathbb{Z}) & \xrightarrow{\iota^*} & \overline{H}^{n-1}(B, \mathbb{Z}) \\
\uparrow & & \uparrow \\
\end{array}
$$

$$\cdots \longrightarrow H_1(X \backslash B; \mathbb{Z}) \xrightarrow{j_*} H_1(X, \mathbb{Z}) \xrightarrow{p_*} H_1(X, X \backslash B, \mathbb{Z}) \longrightarrow H_0(X \backslash B, \mathbb{Z}).$$

By commutativity of the diagram above, and by taking into account the exactness of the pair $(X, X \backslash B)$ in homology, the restriction $\iota^*(\zeta)$ of ζ to B is dual to $p_*([\gamma])$, where $p_* \colon H_1(X, \mathbb{Z}) \to H_1(X, X \backslash B, \mathbb{Z})$. By contradiction we assume that $p_*([\gamma]) = 0$; thus $p_*([\gamma]) = j_*(\beta)$, for some $\beta \in H_1(X, \mathbb{Z})$ and for $j \colon X \backslash B \hookrightarrow X$. Since $X \backslash B$ is connected, the Hurewicz homomorphism $\Pi \colon \pi_1(X \backslash B, x_0) \to H_1(X \backslash B, \mathbb{Z})$ is surjective. Thus $\beta = g_*([S^1])$ where $g \colon (S^1, s_0) \to (X \backslash B, x_0)$. Let $\bar{g} \colon (S^1, s_0) \to (X, x_0)$ be defined by $\bar{g} = j \circ g$. Thus the composition $\overline{h}^{\mathbb{C}} := (\bar{g} \times id) \circ \widetilde{h}^{\mathbb{C}} \colon S^1 \times E^{\mathbb{C}} \to \mathbb{R}$ is a continuous family of self-adjoint Fredholm maps parameterized by S^1. By definition of the spectral flow for Fredholm Hermitian forms, we have that $\mathrm{sf}(\widetilde{h}^{\mathbb{C}}, S^1) = \mathrm{sf}(L^{\mathbb{C}}, S^1)$ where $L^{\mathbb{C}} \colon S^1 \to \mathscr{F}_S(E^{\mathbb{C}})$ is the smooth path of self-adjoint Fredholm operators representing the family $\widetilde{h}$ with respect to the scalar product of $E^{\mathbb{C}}$. Moreover since $\bar{g}(S^1)$ does not intersect B then $\mathrm{sf}(L^{\mathbb{C}}, S^1) = 0$.

In fact, by Lemma 1.11, $\mathrm{sf}(L^{\mathbb{C}}, S^1) = \mathrm{sf}(L, S^1) \neq 0$, and [4, Theorem 1] ensures the existence of a critical point $u \neq 0$ of some $f_\lambda, \lambda \in S^1$, in every neighborhood V of 0. In fact, it is enough to choose $u_n \neq 0$ in $V_n = B(0, 1/n)$ and to assume (by passing, if necessary, to subsequences) that the corresponding sequence λ_n converges to some $\lambda_* \in S^1$. Let $x_n = \psi_{\lambda_n}(u_n)$. By construction of ψ each x_n is a critical point of f_{λ_n} not belonging to $\sigma(S^1)$ and $x_n \to \sigma(\lambda_*)$. Thus λ_* is a bifurcation point.

Now by Proposition 1.10, we have

$$
\begin{aligned}
\mathrm{sf}(L \circ g, S^1) &= \mathrm{sf}(L^{\mathbb{C}} \circ g, S^1) = \bar{c}_1\left(\mathrm{Ind}_\alpha(L^{\mathbb{C}} \circ g)\right)[S^1] = \langle \bar{g}^*(\bar{c}_1\left(\mathrm{Ind}_\alpha L^{\mathbb{C}}\right)), [S^1]\rangle \\
&= \langle \bar{c}_1\left(\mathrm{Ind}_\alpha L^{\mathbb{C}}\right), \bar{g}_*([S^1])\rangle = \langle \bar{c}_1\left(\mathrm{Ind}_\alpha L^{\mathbb{C}}\right), j_* g_*([S^1])\rangle = \\
&= \langle c, \gamma \rangle \neq 0,
\end{aligned}
$$

which is a contradiction. Therefore $p_*([\gamma]) \neq 0$ and hence ζ restrict to a nontrivial class in $\overline{H}^{n-1}(B, \mathbb{Z})$. Now the conclusion follows by taking into account the homological characterization of the Lebesgue covering dimension (see for instance [5, Chapter VIII, Theorem VIII.4]) of a topological space. $\hfill q.e.d.$

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